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ON THE SINGULARITIES OF CONTACT FORCES IN THE BENDING OF PLATES WITH FINE INCLUSIONS*

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A number of problems on the bending of plates with thin inclusions that are distinguished by the conditions at the inclusions is examined. The problems are reduced to systems of integral equations whose characteristic part has the following form in the general case

$$L\varphi \equiv \int_{-1}^{1} \frac{(t-\tau)^2}{2}\left[a\frac{\operatorname{sgn}(t-\tau)}{2} + \frac{b}{\pi i}\ln\frac{1}{|t-\tau|}\right]\varphi(\tau)\,d\tau = f(t) \tag{0.1}$$

Special cases of equations with the characteristic part (0.1) have been examined earlier in /1-4 /, where the solution $\varphi(\tau)$ was sought,in the class of functions with non-integrable singularities by using the apparatus of regularization of divergent integrals /5/. It should be noted that V.M. Tolkachev first drew attention to the absence of solutions with integrable singularities in such problems. An exact solution of (0.1) is constructed below using the method employed in /6/, which provides a rigorous foundation for the approach utilized in /1-4/, and also enables us to find the exact form of the singularities for the problems considered in this paper.

1. Formulas for the limit values of the fundamental quantities on a slit.

We consider a rectangular $(|x| \leqslant a_1 = a/2,\ 0 \leqslant y \leqslant b)$ hinge-supported plate within which there is a defect on the segment $y = l = b/2,\ |x| < c_1 = c/2$ that causes a break in the continuity of the fundamental quantities /1-3/

$$\langle w\rangle = \omega(x),\ \langle w_y'\rangle = \chi(x),\ \langle M_y\rangle = \mu(x),\ \langle V_y\rangle = \psi(x) \tag{1.1}$$
$$\langle f\rangle = f(x,\ l-0) - f(x,\ l+0),\quad \omega = \chi = \mu = \psi \equiv 0 \quad \text{for}$$
$$c_1 < |x| < a_1$$

A thin absolutely rigid inclusion can be such a defect, for instance. To simplify the discussion, we will consider the load applied only to the inclusion, and the plate deflection $w(x, y)$ obtained during settling of the inclusion to be even in x. Then the function satisfying the equation $\Delta^2 w = 0$ for $y \neq l$ and the boundary conditions $w = M_x = 0\ (x = \pm a_1)$, $w = M_y = 0\ (y = 0, b)$, can be represented in the form

$$w(x,y) = \sum_{k=1,3,5,\ldots}^{\infty} \cos\alpha x Y_k(y),\quad \alpha = \alpha_k = \pi a^{-1}k \tag{1.2}$$
$$Y_k(y) = A_k \operatorname{sh}\alpha y + B_k\,\alpha y \operatorname{ch}\alpha y \quad (0 \leqslant y < l)$$
$$Y_k(y) = C_k \operatorname{sh}\alpha(b-y) + D_k\alpha(b-y)\operatorname{ch}\alpha(b-y)\ (l < y \leqslant b)$$

Realization of the conditions (1.1) results in the expression of A_k, B_k, C_k, D_k in terms of ω, χ, μ, ψ. As is shown in /3/, it is more convenient here to go over to the functions

$$\psi_0(\xi) = \omega'''(\xi),\ \psi_1(\xi) = \chi''(\xi),\ \psi_2(\xi) = \mu'(\xi),\ \psi_3(\xi) = \psi(\xi) \tag{1.3}$$

which results in identical principal parts of the kernels in the systems of integral equations obtained below. Obviously, the functions ψ_1 and ψ_3 are even while ψ_0 and ψ_2 are odd.

Representation (1.2) enables the limit values for w, w_y', M_y and V_y to be found easily for $y = l \pm 0$. As in /3/, it turns out to be more convenient to go over to the primitives of the quantities mentioned. We have

$$w(x, l\pm 0) = \sum_{k=1,3,5,\ldots}^{\infty} \int_{-c_1}^{c_1} \Big[\pm\frac{\sin\alpha(x-\xi)}{a\alpha^3}\psi_0(\xi) - \tag{1.4}$$
$$\frac{\cos\alpha(x-\xi)}{ia\alpha^3}(s_1\varkappa_{01}i\psi_1(\xi) - ir_1\varkappa_{03}\psi_3(\xi))\Big]d\xi$$

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$$i\int w_y'\,dx = \sum_{k=1,3,5,\ldots}^{\infty}\int_{-c_1}^{c_1}\Big[\pm\frac{\sin\alpha(x-\xi)}{a\alpha^3}\,i\psi_1(\zeta) -$$
$$\frac{\cos\alpha(x-\xi)}{ia\alpha^3}\,(s_1\varkappa_{10}\psi_0(\xi)+r_1\varkappa_{12}\psi_2(\xi))\Big]\,d\xi$$
$$i\int\int M_y(dx)^2 = \sum_{k=1,3,5,\ldots}^{\infty}\int_{-c_1}^{c_1}\Big[\pm\frac{\sin\alpha(x-\xi)}{a\alpha^3}\,i\psi_2(\xi) -$$
$$\frac{\cos\alpha(x-\xi)}{a\alpha^3}\,(-t_1\varkappa_{21}\psi_1(\xi)-s_1\varkappa_{23}\psi_3(\xi))\Big]\,d\xi$$
$$\int\int\int V_y(dx)^3 = \sum_{k=1,3,5,\ldots}^{\infty}\int_{-c_1}^{c_1}\Big[\pm\frac{\sin\alpha(x-\xi)}{a\alpha^3}\,\psi_3(\xi) -$$
$$\frac{\cos\alpha(x-\xi)}{a\alpha^3}\,(it_1\varkappa_{30}\psi_0(\xi)-s_1\varkappa_{32}i\psi_2(\xi))\Big]\,d\xi$$
$$\varkappa_{01}=\varkappa_{23}=\mathrm{th}\,\rho+\frac{1-\nu}{1+\nu}\,\frac{\rho}{\mathrm{ch}^2\rho},\quad \varkappa_{03}=\mathrm{th}\,\rho-\frac{\rho}{\mathrm{ch}^2\rho}$$
$$\varkappa_{21}=\mathrm{th}\,\rho+\frac{1-\nu}{3+\nu}\,\frac{\rho}{\mathrm{ch}^2\rho},\quad \varkappa_{10}=\varkappa_{32}=\mathrm{cth}\,\rho+\frac{1-\nu}{1+\nu}\,\frac{\rho}{\mathrm{sh}^2\rho}$$
$$\varkappa_{12}=\mathrm{cth}\,\rho-\frac{\rho}{\mathrm{sh}^2\rho},\quad \varkappa_{30}=\mathrm{cth}\,\rho+\frac{1-\nu}{3+\nu}\,\frac{\rho}{\mathrm{sh}^2\rho}$$
$$s_1=\frac{1+\nu}{2},\quad r_1=\frac{1}{2D},\quad t_1=\frac{(1-\nu)(3+\nu)}{2}\,D$$
$$\rho=\rho_k=\frac{1}{2}\pi b a^{-1}k$$

Let us isolate the principal parts of the integral operators obtained in (1.4). To do this, the asymptotic form $\varkappa_{ij}=1+O(\rho_k e^{-2\rho_k})$ should be taken into account and the following formulas should be used:

$$\sum_{k=1,3,5,\ldots}^{\infty}\frac{\sin kt}{k^3}=-\frac{1}{8}\pi t^2\,\mathrm{sgn}\,t+\frac{1}{8}\pi^2 t \tag{1.5}$$
$$\sum_{k=1,3,5,\ldots}^{\infty}\frac{\cos kt}{k^3}=-\frac{t^2}{4}\Big(\ln\frac{1}{|t|}+\frac{3}{2}+\ln 2\Big)+\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k^3}+$$
$$\sum_{n=1}^{\infty}\frac{2^{2n-1}-1}{(2n+2)!\,2n}\,B_n t^{2n+2}\quad(|t|<\pi)$$

Formulas (1.5) are obtained by triple integration of the equations (/7/, 415.06)

$$\sum_{k=1,3,5,\ldots}^{\infty}e^{ikz}=-\frac{1}{2i\sin z}=-\frac{1}{2i}\Big(\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2(2^{2n-1}-1)}{(2n)!}\,B_n z^{2n-1}\Big)$$

with the constant of integration determined from the asymptotic form as $z\to 0$ and separation into real and imaginary parts in the equation obtained. It should be noted that the series on the left in the initial equation is divergent and is Abel-summed. Term-by-term integration of the series is hence allowable and results in a convergent series.

Taking (1.5) into account we rewrite (1.4) in the form

$$w(x,l\pm 0)=\int_{-c_1}^{c_1}\frac{(x-\xi)^2}{4}\Big[\mp\frac{\mathrm{sgn}(x-\xi)}{2}\,\psi_0(\xi)+ \tag{1.6}$$
$$\frac{1}{\pi i}\ln\frac{1}{|x-\xi|}(s_1 i\psi_1(\xi)-ir_1\psi_3(\xi))\Big]\,d\xi+R_0(x)$$
$$i\int w_y'\,dx=\int_{-c_1}^{c_1}\frac{(x-\xi)^2}{4}\Big[\mp\frac{\mathrm{sgn}(x-\xi)}{2}\,i\psi_1(\xi)+$$
$$\frac{1}{\pi i}\ln\frac{1}{|x-\xi|}(s_1\psi_0(\xi)+r_1\psi_2(\xi))\Big]\,d\xi+R_1(x)$$
$$i\int\int M_y(dx)^2=\int_{-c_1}^{c_1}\frac{(x-\xi)^2}{4}\Big[\mp\frac{\mathrm{sgn}(x-\xi)}{2}\,i\psi_2(\xi)+$$
$$\frac{1}{\pi i}\ln\frac{1}{|x-\xi|}(-t_1\psi_1(\xi)-s_1\psi_3(\xi))\Big]\,d\xi+R_2(x)$$
$$\int\int\int V_y(dx)^3=\int_{-c_1}^{c_1}\frac{(x-\xi)^2}{4}\Big[\mp\frac{\mathrm{sgn}(x-\xi)}{2}\,\psi_3(\xi)+$$
$$\frac{1}{\pi i}\ln\frac{1}{|x-\xi|}(it_1\psi_0(\xi)-s_1 i\psi_2(\xi))\Big]\,d\xi+R_3(x)$$

where integral operators with infinitely differentiable kernels are included in $R_m(x)$. We note that (1.6) is fairly general in nature since a change in the boundary conditions and the shape of the plate itself will result in a change in only $R_m(x)$.

2. Formulation of the problem and derivation of the integral equations.

Different kinds of defects are defined by combinations of boundary conditions on the slit edges $y = l \pm 0$, $|x| < c_1$. Starting from the fundamental boundary conditions of the theory of plate bending: clamping, support, and a free edge and considering the case in which the function $\psi_3(\xi)$ is unknown, we obtain the following problem on the bending of plates with thin inclusions.

Problem 1.1. We consider both edges of the slit in a plate connected rigidly to the inclusion i.e., the conditions of plate clamping in the inclusion $w = W_0(x)$, $w_y' = W_1(x)$ are satisfied for $y = l \pm 0$, $|x| < c_1$. In this case $\psi_0(\xi) \equiv \psi_1(\xi) \equiv 0$ and the problem reduces to a system of two integral equations in $\psi_2(\xi)$, $\psi_3(\xi)$

$$-\int_{-c_1}^{c_1} \frac{(x-\xi)^2}{2\pi} \ln \frac{1}{|x-\xi|} \psi_3(\xi)\, d\xi = 4D(W_0(x) - R_0(x)) \tag{2.1}$$

$$-\int_{-c_1}^{c_1} \frac{(x-\xi)^2}{2\pi} \ln \frac{1}{|x-\xi|} i\psi_2(\xi)\, d\xi = 4D\Big(i \int W_1(x)\, dx - R_1(x)\Big) \tag{2.2}$$

Problem 1.2. The slit edge $y = l - 0$ is rigidly connected to the inclusion and the clamping conditions $w = W_0(x)$, $w_y' = W_1(x)$ are satisfied on it. A break in the plate occurred along the edge $y = l + 0$ and between the plate and the inclusion a hinge was formed. Support conditions $w = W_0(x)$, $M_y = M(x)$ are satisfied on this edge. Then $\psi_0(\xi) \equiv 0$ and the problem reduces to the system

$$\int_{-c_1}^{c_1} \frac{(x-\xi)^2}{4\pi} \ln \frac{1}{|x-\xi|} (s_1\psi_1(\xi) - r_1\psi_3(\xi))\, d\xi = W_0(x) - R_0(x) \tag{2.3}$$

$$\int_{-c_1}^{c_1} \frac{(x-\xi)^2}{4} \Big[\frac{\operatorname{sgn}(x-\xi)}{2} i\psi_1(\xi) + \frac{1}{\pi i} \ln \frac{1}{|x-\xi|} r_1\psi_2(\xi)\Big] d\xi = i\int W_1(x)\, dx - R_1(x) \tag{2.4}$$

$$\int_{-c_1}^{c_1} \frac{(x-\xi)^2}{4} \Big[-\frac{\operatorname{sgn}(x-\xi)}{2} i\psi_2(\xi) - \frac{1}{\pi i} \ln \frac{1}{|x-\xi|} (t_1(\xi) + s_{13}\ \xi))\Big] d\xi = i \int\int M(x)(dx)^2 - R_2(x) \tag{2.5}$$

Eq.(2.3) is of the same type as (2.1) and (2.2). Executing the combination $-s_1$ (2.3), $-i$ (2.4), $+ir_1$ (2.5), we arrive at the equation

$$\int_{-c_1}^{c_1} \frac{(x-\xi)^2}{4} \Big(\frac{\operatorname{sgn}(x-\xi)}{2} - \frac{i}{\pi i} \ln \frac{1}{|x-\xi|}\Big) H(\xi)\, d\xi = f(x) \tag{2.6}$$

$$H(\xi) = \psi_1(\xi) + r_1\psi_2(\xi)$$

$\psi_1(\xi)$ and $\psi_2(\xi)$ are found as even and odd components of the functions $H(\xi)$, while $\psi_3(\xi)$ is found from (2.3) after determining $\psi_1(\xi)$.

Problem 1.3. The slit edge $y = l - 0$ is rigidly connected to the inclusion, and clamping conditions $w = W_0(x)$, $w_y' = W_1(x)$ are satisfied on it. Delamination of the plate from the inclusion occurred along the edge $y = l + 0$ and a crack formed. Free edge conditions $M_y = M(x)$, $V_y = V(x)$ are satisfied on this edge. This problem was examined in /3/ where it is reduced to the integral equation

$$\int_{-c_1}^{c_1} \frac{(x-\xi)^2}{4} \Big(\frac{\operatorname{sgn}(x-\xi)}{2} + \frac{s_1 + i\sqrt{r_1 t_1}}{\pi i} \ln \frac{1}{|x-\xi|}\Big) H(\xi)\, d\xi = f(x) \tag{2.7}$$

$$H(\xi) = \psi_0(\xi) + i\psi_1(\xi) - r_1^{1/2} t_1^{-1/2} (i\psi_2(\xi) + \psi_3(\xi))$$

Problem 2.2. A break in the plate occurred on both edges of the slit and the support conditions are satisfied. In this case $\psi_0(\xi) = \psi_2(\xi) \equiv 0$ and the system of equations in $\psi_1(\xi)$, $\psi_3(\xi)$ is analogous to system (2.1) and (2.2).

Problem 2.3. A beak occurred on the edge $y = l - 0$ and the support conditions are satisfied, delamination occurred on the edge $y = l + 0$ and the free edge conditions are satisfied. The problem reduces to a system analogous to (2.3)-(2.5).

3. The case of an inclusion located on the boundary. We consider a rectangular plate $(|x|\leqslant a_1, 0\leqslant y\leqslant l=b/2)$, hinge-supported on three sides $x=\pm a_1$ and $y=0$. The side $y=l$ adheres to a rigid rod, which results in mixed boundary conditions

$$w(x, l-0)=W_0(x),\ w_y'(x, l-0)=W_1(x)\quad (|x|<c_1) \tag{3.1}$$

$$M_y(x, l-0)=0,\ V_y(x, l-0)=0\quad (c_1<|x|<a_1) \tag{3.2}$$

We use the representation (1.2) for $0\leqslant y<l$. Introducing the unknown functions $\psi_3(x)=V_y(x, l-0), \psi_2(x)=\mu'(x), \mu(x)=M_y(x, l-0)$, for $|x|<c_1$, we express A_k, B_k in terms of them. Realizing conditions (3.1), we arrive at the system of integral equations

$$\sum_{k=1,3,5,\ldots}^{\infty}\int_{c_1}^{c_1}\left[-\frac{\sin\alpha(x-\xi)}{a\alpha^3t_1}\lambda_{02}\psi_2(\xi)+\frac{\cos\alpha(x-\xi)}{a\alpha^3t_1}\lambda_{03}\psi_3(\xi)\right]d\xi=W_0(x) \tag{3.3}$$

$$\sum_{i=1,3,5,\ldots}^{\infty}\int_{-c_1}^{c_1}\left[\frac{\sin\alpha(x-\xi)}{a\alpha^3t_1}\lambda_{13}\psi_3(\xi)+\frac{\cos\alpha(x-\xi)}{a\alpha^3t_1}\lambda_{12}\psi_2(\xi)\right]d\xi=$$
$$\int W_1(x)\,dx$$

$$\lambda_{02}=\frac{(1+\nu)(3+\nu)}{2\,\mathrm{th}^2\rho+1+\nu},\quad \lambda_{03}=\lambda_{12}=\frac{2(3+\nu)\,\mathrm{th}\,\rho}{2\,\mathrm{th}^2\rho+1+\nu}$$

$$\lambda_{13}=(3+\nu)\frac{2\,\mathrm{th}^2\rho-1+\nu}{2\,\mathrm{th}^2\rho+1+\nu}$$

which can be rewritten in the form

$$\int_{-c_1}^{c_1}\frac{(x-\xi)^2}{4t_1}\left[(1+\nu)\frac{\mathrm{sgn}(x-\xi)}{2}\psi_2(\xi)-\frac{2}{\pi}\ln\frac{1}{|x-\xi|}\psi_3(\xi)\right]d\xi= \tag{3.4}$$
$$W_0(x)-R_0(x)$$

$$\int_{-c_1}^{c_1}\frac{(x-\xi)^2}{4t_1}\left[-(1+\nu)\frac{\mathrm{sgn}(x-\xi)}{2}\psi_3(\xi)-\right. \tag{3.5}$$
$$\left.\frac{2}{\pi}\ln\frac{1}{|x-\xi|}\psi_2(\xi)\right]d\xi=\int W_1(x)\,dx-R_1(x)$$

Forming the combination i (3.4)-(3.5), we arrive at one equation

$$\int_{-c_1}^{c_1}\frac{(x-\xi)^2}{4t_1}\left[(1+\nu)\frac{\mathrm{sgn}(x-\xi)}{2}+\frac{2}{\pi i}\ln\frac{1}{|x-\xi|}\right]H(\xi)\,d\xi=f(x) \tag{3.6}$$
$$H(\xi)=i\psi_2(\xi)+\psi_3(\xi)$$

It is seen that Eq.(0.1) is characteristic for (2.1), (2.2), (2.6), (2.7), (3.6), and therefore, the form of the singularities of the functions $\psi_i(\xi), H(\xi)$ as $\xi\to\pm c_1$ will be the same as the function $\varphi(\tau)$ as $\tau\to\pm1$.

We will now construct the exact solution of (0.1) which enables these singularities to be isolated in explicit form.

4. Cauchy-type integrals with density having a non-integrable singularity. We write the density in the form $\varphi(\tau)=(1-\tau)^{\mu}(1+\tau)^{\nu}\varphi_0(\tau)$, where $\varphi_0(\tau)\in H$ (satisfies the Hölder condition), $\varphi_0(\pm1)\neq0$. We let H^* denote the class of functions with integrable singularities, i.e., those for which $\mathrm{Re}(\mu,\nu)>-1$, and H^{**} the class of functions for which at least one of the numbers $\mathrm{Re}\,\mu, \mathrm{Re}\,\nu$ is not greater than -1.

Assuming that $f'''(t)\in H$ and differentiating (0.1) three times with respect to t, we have

$$a\varphi(t)+\frac{b}{\pi i}\int_{-1}^{1}\frac{\varphi(\tau)}{\tau-t}\,d\tau=f'''(t)\quad (|t|<1) \tag{4.1}$$

The solution of this equation in the class H^* is known /6, 8/; however, it cannot be the solution of (0.1) giving a function differing from $f(t)$ by a polynomial of second degree on the left-hand side of (0.1) in the formulation. In order for this polynomial to vanish it is necessary to have three arbitrary constants in the solution of (4.1), while the homogeneous equation corresponding to (4.1) has only one linearly independent solution in the class H^*. This reasoning indeed prescribes the emergence into the broader class H^{**}, where as is shown below, any necessary number of linearly independent solutions of the homogeneous equation can be obtained.

In order to carry the scheme of the solution of (4.1) used in /6, 8/ over to the case

$\varphi(\tau) \in H^{**}$, we shall understand the Cauchy-type integral

$$\Phi(z)=\frac{1}{2\pi i}\int_{-1}^{1}\frac{\varphi(\tau)\,d\tau}{\tau-z}=\frac{1}{2\pi i}\int_{-1}^{1}\frac{(1-\tau)^{\mu}(1+\tau)^{\nu}\varphi_0(\tau)}{\tau-z}\,d\tau \qquad (4.2)$$
$$z \,\overline{\in}\, [-1,1]$$

in the regularized sense /5/ and the integral in the Cauchy principal value sense

$$\Phi(t)=\frac{1}{2\pi i}\int_{-1}^{1}\frac{\varphi(\tau)\,d\tau}{\tau-t}=\frac{1}{2\pi i}\int_{-1}^{1}\frac{(1-\tau)^{\mu}(1+\tau)^{\nu}\varphi_0(\tau)}{\tau-t}\,d\tau,\ t\in(-1,1) \qquad (4.3)$$

It can be shown* (*Gribnyak S.T., Onishchuk O.V. and Farshait, P.G., Solution of an integral equation with a principal kernel in the class of functions with non-integrable singularities. Dep. in UkrNIINTI, No.1199, Odessa, July 11, 1984) that the function $\Phi(z)$ is here analytic outside the contour $[-1,1]$, behaves as $O((1-z)^{\mu_*})$ as $z\to 1$ ($\mu_* = \min(\operatorname{Re}\mu, 0)$), $O((1+z)^{\nu_*})$ as $z=\to -1$ ($\nu_* = \min(\operatorname{Re}\nu, 0)$), $O(z^{-1})$ as $z\to\infty$ and has limit values on $(-1,1)$ for which the Sokhotskii formulas are satisfied

$$\Phi^+(t)-\Phi^-(t)=\varphi(t),\quad \Phi^+(t)+\Phi^-(t)=2\Phi(t) \qquad (4.4)$$

5. Solution of the Riemann problem and singular integral equations in the class with non-integrable singularities. We consider the equation

$$a\varphi(t)+\frac{b}{\pi i}\int_{-1}^{1}\frac{\varphi(\tau)}{\tau-t}\,d\tau=s(t)\quad (|t|<1) \qquad (5.1)$$

We seek the solution in the class of functions $\varphi(t)=(1-t)^{\mu}(1+t)^{\nu}\varphi_0(t)$, where μ,ν satisfy the conditions

$$-m-1<\operatorname{Re}\mu<-m,\ -n-1<\operatorname{Re}\nu<-n \qquad (5.2)$$

and m, n are integers. Following /6/, (p.176, 482) we consider the function (4.2) where $\varphi(\tau)$ is the solution of (5.1). Then on the basis of (4.4), $\Phi(z)$ will be a solution of the Riemann problem

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t)\quad(-1<t<1),\ \Phi(\infty)=0 \qquad (5.3)$$
$$\Phi(z)=O((1-z)^{\mu_*})\ \text{as}\ z\to 1,\ \Phi(z)=O((1+z)^{\nu_*})\ \text{as}\ z\to -1$$
$$G(t)=\frac{a-b}{a+b}=G=\text{const},\quad g(t)=\frac{s(t)}{a+b}$$

The inverse can also be proved (see the footnote): for any solution of problem (5.3) the function $\varphi(\tau)=\Phi^+(\tau)-\Phi^-(\tau)$ will be a solution of Eq.(5.1), where taking $m\geqslant 1, n\geqslant 1$ in (5.2) we obtain the solution $\varphi(\tau)$ in the class H^{**}.

Following /6/,Sect.43, /8/,Sect.26, we consider the function

$$X(z)=(1-z)^{-\beta-m}(1+z)^{\beta-n-1},\ \beta=-\frac{\ln G}{2\pi i},\ \arg G\in(-2\pi,0) \qquad (5.4)$$

which is analytic in a plane with the slit $[-1,1]$, satisfies the homogeneous condition $X^+(t)=GX^-(t)$ and behaves as $O((1-z)^{\mu})$ as $z\to 1$, $O((1+z)^{\nu})$ as $z\to -1$, and $O(z^{-\varkappa})$ as $z\to\infty$ (μ and ν satisfy conditions (5.2), $\varkappa=m+n+1$). Then the function

$$F(z)=\frac{\Phi(z)}{X(z)}-\Psi(z),\quad \Psi(z)=\frac{1}{2\pi i}\int_{-1}^{1}\frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-z} \qquad (5.5)$$

is regular in the whole z plane with the exception, perhaps, of the points ± 1. We consider two cases used in solving (0.1).

Case 1. If $m\geqslant -1, n\geqslant -1$ in (5.2), then the points $z=\pm 1$ will be eliminable singularities for the function $F(z)$ and by Liouville's theorem $F(z)=P_{\varkappa-1}(z)$ is a polynomial of degree $\varkappa-1$. The the solution of problem (5.3) has the form

$$\Phi(z)=X(z)(\Psi(z)+P_{\varkappa-1}(z)) \qquad (5.6)$$

and contains $\varkappa$ arbitrary constants.

The value $\varkappa=3$ required in Sect.4 can be obtained by taking any m and $n=2-m$ in (5.2). However, the condition of convergence of the energy integral of a bent plate imposed from mechanical considerations results in the single set $m=n=1$. It can be shown (see the previous footnote) that the condition of plate clamping at the ends of the inclusion (the continuity of w, w'_x, w_y' at these points) is also satisfied here.

Using (4.4) and (5.6), we find the solution of (4.1) in the class H^{**} for which $-2<\operatorname{Re}\mu<-1,\ -2<\operatorname{Re}\nu<-1$:

$$\varphi(t)=\frac{af'''(t)}{a^2-b^2}-\frac{b}{\pi i}\frac{X_1^+(t)}{a^2-b^2}\int_{-1}^{1}\frac{f'''(\tau)}{X_1^+(\tau)}\frac{d\tau}{\tau-t}+X_1^+(t)\sum_{k=0}^{2}a_kt^k \tag{5.7}$$

$$X_1^+(t)=(1-t)^{-1-\beta}(1+t)^{-2+\beta}$$

where a_k are constants determined later from system (6.5) obtained on substituting $\varphi(t)$ into (0.1).

Case 2. On substituting (5.7) into (0.1) below, the question will arise of the solution of (5.1) in the class of functions for which

$$1<\operatorname{Re}\mu<2,\ 1<\operatorname{Re}\nu<2,\ m=n=-2,\ \varkappa=-3 \tag{5.8}$$

and about the conditions imposed on $s(t)$ for solvability in this class.

In the case under consideration, the function (5.5) can have poles of multiplicity one at the points $z=\pm 1$; then the solution of problem (5.3) and Eq.(5.1) will take the form

$$\Phi(z)=X_2(z)\left(\Psi(z)+\frac{B_1}{1-z}+\frac{B_2}{1+z}\right) \tag{5.9}$$

$$X_2(z)=(1-z)^{2-\beta}(1+z)^{1+\beta}$$

$$\varphi(t)=\frac{X_2^+(t)}{a^2-b^2}\left(\psi(t)+\frac{C_1}{1-t}+\frac{C_2}{1+t}\right) \tag{5.10}$$

$$X_2^+(t)=(1-t)^{2-\beta}(1+t)^{1+\beta}$$

$$\psi(t)=a\frac{s(t)}{X_2^+(t)}-\frac{b}{\pi i}\int_{-1}^{1}\frac{s(\tau)}{X_2^+(\tau)}\frac{d\tau}{\tau-t},\quad C_k=-2(a+b)bB_k$$

under the following necessary and sufficient conditions associated with the requirement $\Phi(\infty)=0$:

$$\int_{-1}^{1}s(\tau)[X_2^+(\tau)]^{-1}\tau^j d\tau+B_1-(-1)^jB_2=0\qquad (j=0,1,2) \tag{5.11}$$

In general, if $s(t)$ satisfies just the Hölder condition, then as $t\to\pm 1$ $\psi(t)$ can become of an order higher than the first at infinity, and conditions (5.8) will not be satisfied for any selection of C_1 and C_2. Below $s(t)=f(t)$ and $s'''(t)\in H$. Then by the Taylor formula $s(t)=P_2^{\pm}(t)+o((1\mp t)^2)$ in the neighbourhood of the points $t=\pm 1$. The function $\psi_0(t)=[X_2^+(t)]^{-1}P_2(t)$ is the solution of the homogeneous equation

$$a\psi_0(t)-\frac{b}{\pi i}\int_{-1}^{1}\frac{\psi_0(\tau)}{\tau-t}d\tau=0 \tag{5.12}$$

which is conjugate to (5.1) $f'''(t)\equiv 0$ in (5.7) and β is replaced by $1-\beta$, $X_1^+(t)$ by $[X_2^+(t)]^{-1}$), consequently, the function $\psi(t)$ will be bounded as $t\to\pm 1$. Setting $C_1=C_2=0$, we obtain that when the condition

$$\int_{-1}^{1}s(\tau)[X_2^+(\tau)]^{-1}\tau^j d\tau=0\quad (j=0,1,2) \tag{5.13}$$

is satisfied, Eq.(5.1) has a solution satisfying (5.8). Obviously (5.13) is the condition of orthogonality of the right-hand side of (5.1) to the solutions of Eq.(5.12).

6. Solution of the integral equation (0.1). We write (5.7) in the form

$$\varphi(t)=\frac{d^3g_3(t)}{dt^3}+X_1^+(t)\sum_{k=0}^{2}a_kt^k \tag{6.1}$$

$$g_3(t)=af(t)-\frac{b}{\pi i}X_2^+(t)\int_{-1}^{1}\frac{f(\tau)}{X_2^+(\tau)}\frac{d\tau}{\tau-t}$$

We have here used the formula

$$\frac{d}{dt}\left[\sigma_n(t)\int_{-1}^{1}\frac{f(\tau)}{\sigma_n(\tau)}\frac{d\tau}{\tau-t}\right]=\sigma_{n-1}(t)\left[\int_{-1}^{1}\frac{f'(\tau)}{\sigma_{n-1}(\tau)}\frac{d\tau}{\tau-t}+\right.$$

$$\left.(2n-2)\int_{-1}^{1}\frac{f(\tau)}{\sigma_n(\tau)}d\tau\right],\quad \sigma_n(t)=(1-t)^{n-\beta}(1+t)^{n-1+\beta},$$

$$n=2,1,0$$

which is confirmed by taking into account (4.16) /6/. Formula (4.16) /6/ as well as the formulas (3.14), (3.15) /6/ used in its derivation hold even in the case when $\varphi'(t)\in H^{**}$ if formula (1.6) /1/ is taken into account.

We substitute (6.1) into the left-hand side of (0.1)

$$L\varphi = Lg_3''' + \sum_{k=0}^{2} a_k Q_k(t), \quad Q_k(t) = L[X_1^+(t)\, t^k] \tag{6.2}$$

Using formula (1.6) /1/ and (A.6.7) /2/, we calculate $Q_k(t) = b_{2k}t^2 + b_{1k}t + b_{0k}$

$$Q_0(t) = \frac{1+\delta}{2}t^2 - \gamma t - \frac{\alpha - 1 - \delta}{2}$$

$$Q_1(t) = \frac{\gamma}{2}t^2 - \left(\frac{3}{2} - \alpha + \delta\right)t + \frac{1-2\beta}{2}(2 + \alpha + \delta)$$

$$Q_2(t) = \left(\frac{\delta - \alpha}{2} + \frac{5}{4}\right)t^2 - (1 - 2\beta)\left(\alpha + \frac{3}{2} + \delta\right)t -$$

$$\left(\frac{3}{4} + \frac{(1-2\beta)^2}{4}\right)\alpha + \left(\frac{5}{8} - \frac{3}{8}(1-2\beta)^2 + \frac{\delta}{2}\right)$$

$$\alpha = \frac{1}{2}\pi \operatorname{ctg} \pi\beta - (\ln 2 + \psi(1-\beta) - \psi(1)), \quad \gamma = \frac{1-2\beta}{4\beta(1-\beta)},$$

$$\delta = (1 - 2\beta)\gamma$$

Since $g_3'''(\tau)$ is the solution of (4.1), then

$$Lg_3''' = f(t) - A_2t^2 - A_1t - A_0 \tag{6.3}$$

To find A_0, A_1, A_2 we integrate three times by parts on the left-hand side of (6.3). Taking account of the behaviour of $g_3''(t), g_3'(t), g_3(t)$ as $t \to \pm 1$, and (1.6) /1/, the components outside the integral vanish and (6.3) takes the form

$$ag_3(t) + \frac{b}{\pi i}\int_{-1}^{1} \frac{g_3(\tau)}{\tau - t}\, d\tau = f(t) - \sum_{l=0}^{2} A_l t^l \tag{6.4}$$

Since $g_3(t)$ is the solution of (6.4) in the class (5.8), then the right side of this equation satisfies conditions (5.13) and A_l are found successively from the equations

$$\sum_{l=2-j}^{2} A_l I_{l+j} = \int_{-1}^{1} f(t)[X_2^+(t)]^{-1} t^j\, dt \quad (j = 0, 1, 2)$$

$$I_k = \int_{-1}^{1} [X_2^+(t)]^{-1} t^k\, dt$$

$$I_0 = I_1 = 0, \quad I_2 = -\frac{\pi}{\sin \pi\beta}, \quad I_3 = (1 - 2\beta) I_2,$$

$$I_4 = \left(\frac{3}{2} + \frac{(1-2\beta)^2}{2}\right) I_2$$

Rewriting (6.2) in the form

$$L\varphi = f(t) + \sum_{l=0}^{2}\left(\sum_{k=0}^{2} b_{lk}a_k - A_l\right)t^l$$

we arrive at the conclusion that in order for (5.7) to be a solution of (0.1), it is necessary and sufficient that a_k satisfy the set of equations

$$\sum_{k=0}^{2} b_{lk}a_k = A_l \quad (l = 0, 1, 2) \tag{6.5}$$

Formulas (5.7) and (6.5) yield an exact solution of the characteristic Eq.(0.1). For an approximate solution of the complete equations obtained in Sects.2, 3, it is convenient to use the method of orthogonal polynomials /2/, by writing the desired function in the form

$$\varphi(t) = (1 - t)^{-1-\beta}(1 + t)^{-2+\beta} \sum_{n=0}^{\infty} \varphi_n P_n(t) \tag{6.6}$$

$$P_n(t) = P_n^{-1-\beta, -2+\beta}(t)(n \geqslant 1), \quad P_0(t) = 1 - t^2$$

and using expressions for $Q_n(t)$ for $n \leqslant 2$ and the spectral relationship (see the previous footnote) for $n > 2$

$$L[(1-\tau)^{-1-\beta}(1+\tau)^{-2+\beta}P_n^{-1-\beta, -2+\beta}(\tau)] = -\frac{P_n^{-2+\beta, -1-\beta}(t)}{n(n-1)(n-2)} \tag{6.7}$$

The singularities of the contact forces $\psi_3(\xi)$ at the ends of the defect are determined by the value of β in (5.4), (5.7): for $\xi \to \pm c_1 \mp 0$ $\psi_3(\xi) = O((c_1 \mp \xi)^{-\gamma})$, $\gamma = 1 + \beta$ when $\operatorname{Re} \beta \in [0.5; 1)$, $\gamma = 2 - \beta$ when $\operatorname{Re} \beta \in (0; 0.5]$. Let us mention the values of β for the problems considered above. In Problem 1.1 (total adhesion of the plate to the inclusion) $\beta = \beta_{11} = 0.5$ in Problem 1.2 (a break at one of the edges) $\beta = \beta_{12} = 0.75$, in Problem 1.3 (delamination along one of the edges) $\beta = \beta_{13}$, in the problem in Sect.3 (an inclusion on the boundary) $\beta = \beta_3$

$$\beta_{13} = \frac{3}{4} + \frac{1}{4\pi i}\ln\frac{1-\nu}{3+\nu}, \qquad \beta_3 = \frac{1}{2} + \frac{1}{2\pi i}\ln\frac{1-\nu}{3+\nu}$$

Calculation of the determinant $D(\beta)$ of system (6.5) by using /9/ yields the following values for $\nu = 0.3$, which ensure that the system is solvable $D(\beta_{11}) \approx 0.0694$, $D(\beta_{12}) \approx 0.5876$, $D(\beta_3) \approx 0.1498$, $D(\beta_{13}) \approx -0.3359 + 0.3283i$.

The presence of an imaginary part in β_{13} and β_3 shows that the contact forces in the last two problems contain oscillatory factors in addition to non-integrable singularities.

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ON THE INVERSE PROBLEM OF THE SCATTERING OF ELASTIC WAVES BY A THIN FOREIGN INCLUSION*

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The problem of the remote determination of the shape of an isolated scatterer is considered using longitudinal elastic waves. It is assumed that the scatterer is a thin elastic solid of revolution situated in an elastic space under conditions of rigid contact and that Poisson's ratios of the medium and the scatterer are the same. The use of multifrequency wave information is a special feature of the solution of the problem. The problems of the uniqueness and stability of the solution obtained are also studied.

We mean by the inverse scattering problem the problem of determining the form of the scattering region by analysing the scattered field. The problem, as a rule, is one of a number of ill-posed problems of mathematical physics /1/. The current interest in developments in this direction is caused by the practical needs experienced in such fields as acoustic diagnostics, geophysics, hydroacoustics, medicine, etc. At present several approaches to the study of the form of closed isolated scatterers are known /2—4/. Here the corresponding direct problem of mathematical physics was formulated as a boundary value problem for the Helmholtz equation with Dirichlet boundary conditions and Newmann or impedance boundary conditions on the unknown surface of the body whose location was being determined.

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